

# An Arithmetic Proof of John's Ellipsoid Theorem

Peter M. Gruber and Franz E. Schuster

**Abstract.** Using an idea of Voronoi in the geometric theory of positive definite quadratic forms, we give a transparent proof of John's characterization of the unique ellipsoid of maximum volume contained in a convex body. The same idea applies to the 'hard part' of a generalization of John's theorem and shows the difficulties of the corresponding 'easy part'.

**MSC 2000.** 52A21, 52A27, 46B07.

**Key words.** John's theorem, approximation by ellipsoids, Banach–Mazur distance.

## Introduction and Statement of Results

The following well-known characterizations of the unique ellipsoid of maximum volume in a convex body in Euclidean  $d$ -space are due to John [10] ((i) $\Rightarrow$ (ii)) and Pelczyński [12] and Ball [1] ((ii) $\Rightarrow$ (i)), respectively. For references to other proofs, a generalization and to the numerous applications see [2, 6, 11].

**Theorem 1** *Let  $C \subset \mathbb{E}^d$  be compact, convex, symmetric in the origin  $o$ , and with  $B^d \subset C$ . Then the following claims are equivalent:*

- (i)  $B^d$  is the unique ellipsoid of maximum volume in  $C$ .
- (ii) *There are  $u_k \in B^d \cap \text{bd } C$  and  $\lambda_k > 0, k = 1, \dots, n$ , where  $d \leq n \leq \frac{1}{2}d(d+1)$ , such that*

$$I = \sum_k \lambda_k u_k \otimes u_k.$$

Here,  $B^d$  is the solid unit ball in  $\mathbb{E}^d$ ,  $I$  the  $d \times d$  unit matrix, and for  $u, v \in \mathbb{E}^d$  the  $d \times d$  matrix  $u v^T$  is denoted by  $u \otimes v$ .  $\text{bd}$  stands for boundary.

**Theorem 2** *Let  $C \subset \mathbb{E}^d$  be compact, convex, and with  $B^d \subset C$ . Then the following claims are equivalent:*

- (i)  $B^d$  is the unique ellipsoid of maximum volume in  $C$ .
- (ii) *There are  $u_k \in B^d \cap \text{bd } C$  and  $\lambda_k > 0, k = 1, \dots, n$ , where  $d+1 \leq n \leq \frac{1}{2}d(d+3)$ , such that*

$$I = \sum_k \lambda_k u_k \otimes u_k, \quad o = \sum_k \lambda_k u_k.$$

Our proof of Theorem 1 is based on the idea of Voronoi in the geometric theory of positive definite quadratic forms, to represent ellipsoids in  $\mathbb{E}^d$  with center  $o$  by points in  $\mathbb{E}^{\frac{1}{2}d(d+1)}$ , see [4, 9, 14]. The problem on maximum volume ellipsoids in  $\mathbb{E}^d$  is then transformed into a simple problem on normal cones in  $\mathbb{E}^{\frac{1}{2}d(d+1)}$ , which can be solved easily by Carathéodory's theorem on convex hulls. This idea has been applied before by the first author [8]. The proof of Theorem 2 is a simple extension. The proof of the latter also gives Theorem 4 of Bastero and Romance [3], where  $B^d$  is replaced by a compact connected set with positive measure.

In the context of John's theorem, it is natural to ask whether ellipsoids can be replaced by more general convex or non-convex sets. The following is a slight refinement of results of Giannopoulos, Perissinaki and Tsolomitis [7] and Bastero and Romance [3] (Theorem 3). The result of Giannopoulos et. al. was first observed by Milman in the case, where both bodies are centrally symmetric, see [16].

**Theorem 3** *Let  $C \subset \mathbb{E}^d$  be compact and convex, and  $B \subset C$  compact with positive measure. Then (i) implies (ii), where the claims (i) and (ii) are as follows:*

- (i)  *$B$  has maximum measure amongst all its affine images contained in  $C$ .*
- (ii) *There are  $u_k \in B \cap \text{bd } C$ ,  $v_k \in N_C(u_k)$ , and  $\lambda_k > 0$ ,  $k = 1, \dots, n$ , where  $d+1 \leq n \leq d(d+1)$ , such that*

$$I = \sum_k \lambda_k u_k \otimes v_k, \quad o = \sum_k \lambda_k v_k.$$

Here  $N_C(u)$ ,  $u \in \text{bd } C$ , is the normal cone of  $C$  at  $u$ . For this concept and other required notions and results of convex geometry we refer to [15].

Note that  $B$  is not necessarily unique. A suitable modification of Voronoi's idea applies in the present context and thus leads to a proof of Theorem 3, paralleling our proofs of Theorems 1 and 2. Incidentally, the proof of Theorem 3 shows, why it is *not* clear that property (ii) implies property (i), see the Final Remarks.

### Proof of Theorem 1

For (real)  $d \times d$ -matrices  $A = (a_{ij})$ ,  $B = (b_{ij})$  define  $A \cdot B = \sum a_{ij} b_{ij}$ . The dot  $\cdot$  denotes also the inner product in  $\mathbb{E}^d$ . Easy arguments yield the following:

- (1) Let  $M$  be a  $d \times d$  matrix and  $u, v, w \in \mathbb{E}^d$ . Then  $Mu \cdot v = M \cdot u \otimes v$  and  $(u \otimes v)w = (v \cdot w)u$ .

Next, we specify two tools:

- (2) Each  $d \times d$  matrix  $M$  with  $\det M \neq 0$  can be represented in the form  $M = AR$ , where  $A$  is a symmetric, positive definite and  $R$  is an orthogonal  $d \times d$  matrix.

(Put  $A = (MM^T)^{\frac{1}{2}}, R = A^{-1}M$ , see [5], p.112.) Identify a symmetric  $d \times d$  matrix  $A = (a_{ij})$  with the point  $(a_{11}, \dots, a_{1d}, a_{22}, \dots, a_{2d}, \dots, a_{dd})^T \in \mathbb{E}^{\frac{1}{2}d(d+1)}$ . The set of all symmetric, positive definite  $d \times d$  matrices then is (represented by) an open convex cone  $\mathcal{P} \subset \mathbb{E}^{\frac{1}{2}d(d+1)}$  with apex at the origin. The set

- (3)  $\mathcal{D} = \{A \in \mathcal{P} : \det A \geq 1\}$  is a closed, smooth, strictly convex set in  $\mathcal{P}$  with non-empty interior.

(Use the implicit function theorem and Minkowski's inequality for symmetric, positive definite  $d \times d$  matrices, see [13], p.205.)

(i) $\Rightarrow$ (ii): By (2), any ellipsoid in  $\mathbb{E}^d$  can be represented in the form  $AB^d$ , where  $A \in \mathcal{P}$ . Thus the family of all ellipsoids in  $C$  is represented by the set

$$\mathcal{E} = \{A \in \mathcal{P} : Au \cdot v = A \cdot u \otimes v \leq h_C(v) \text{ for } u, v \in S^{d-1}\},$$

see (1). Here,  $h_C(\cdot)$  is the support function of  $C$ . Clearly,  $\mathcal{E}$  is the intersection of the closed halfspaces

$$(4) \quad \{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} : A \cdot u \otimes v \leq h_C(v)\} : u, v \in S^{d-1},$$

with the set  $\mathcal{P}$ . Thus, in particular,  $\mathcal{E}$  is convex. By (i),  $\mathcal{E} \setminus \{I\} \subset \{A \in \mathcal{P} : \det A < 1\}$ . This, together with (3), shows that

- (5)  $\mathcal{D}$  and  $\mathcal{E}$  are convex,  $\mathcal{D} \cap \mathcal{E} = \{I\}$ , and  $\mathcal{D}$  and  $\mathcal{E}$  are separated by the unique support hyperplane  $\mathcal{H}$  of  $\mathcal{D}$  at  $I$  in  $\mathbb{E}^{\frac{1}{2}d(d+1)}$ .

$\mathcal{E}$  is the intersection of the closed halfspaces in (4) with the set  $\mathcal{P}$ , and these halfspaces vary continuously as  $u, v$  range over  $S^{d-1}$ . Thus the support cone  $\mathcal{K}$  of  $\mathcal{E}$  at  $I$  can be represented as the intersection of those halfspaces, which contain  $I$  on their boundary hyperplanes, i.e. for which  $I \cdot u \otimes v = u \cdot v = h_C(v)$ . Since  $u \cdot v \leq 1$  and  $h_C(v) \geq 1$  and equality holds in both cases precisely when  $u = v \in S^{d-1} \cap \text{bd } C$  (note that  $B^d \subset C$ ), we see that

$$(6) \quad \mathcal{K} = \bigcap_{u \in B^d \cap \text{bd } C} \{A \in \mathbb{E}^{\frac{1}{2}d(d+1)} : A \cdot u \otimes u \leq 1\}.$$

The normal cone  $\mathcal{N}$  of ( $\mathcal{E}$  or)  $\mathcal{K}$  at  $I$  is generated by the exterior normals of these halfspaces,

$$(7) \quad \mathcal{N} = \text{pos} \{u \otimes u : u \in B^d \cap \text{bd } C\}.$$

The cone  $\mathcal{K}$  has apex  $I$  and, by (5), is separated from the convex set  $\mathcal{D}$  by the hyperplane  $\mathcal{H}$ , where  $\mathcal{H}$  is the unique support hyperplane of  $\mathcal{D}$  at  $I$ . By considering the gradient of the function  $A \rightarrow \det A$ , we see that  $I$  is an interior normal vector of  $\mathcal{D}$  at  $I$  and thus a normal vector of  $\mathcal{H}$  pointing away from  $\mathcal{K}$ . Hence  $I \in \mathcal{N}$ . (7) and Carathéodory's theorem for convex cones then yield the following: there are  $u_k \otimes u_k \in \mathcal{N}$ , i.e.  $u_k \in B^d \cap \text{bd } C$ , and  $\lambda_k > 0$  for  $k = 1, \dots, n$ , where  $n \leq \frac{1}{2}d(d+1)$ , such that

$$(8) \quad I = \sum_k \lambda_k u_k \otimes u_k.$$

For the proof that  $n \geq d$ , it is sufficient to show that  $\text{lin}\{u_1, \dots, u_n\} = \mathbb{E}^d$ . If this were not true, we could choose  $u \neq o, u \perp u_1, \dots, u_n$ , and then (1) yields the contradiction

$$0 \neq u^2 = Iu \cdot u = \left( \sum_k \lambda_k (u_k \otimes u_k) u \right) \cdot u = \left( \sum_k \lambda_k (u_k \cdot u) u_k \right) \cdot u = 0.$$

(ii) $\Rightarrow$ (i): Let  $\mathcal{E}$  be as above.  $\mathcal{E}$  is convex.  $B^d \subset C$  implies that  $I$  satisfies all defining inequalities of  $\mathcal{E}$ , in particular those corresponding to  $u = v = u_k, k = 1, \dots, n$ . Since  $h_C(u_k) = 1$ , these inequalities are satisfied even with the equality sign. Thus  $I \in \text{bd } \mathcal{E}$ . Define  $\mathcal{K}, \mathcal{N}$  and  $\mathcal{H}$  as before. (ii) implies that  $I \in \mathcal{N}$ . Hence  $\mathcal{K}$  is contained in the closed halfspace with boundary hyperplane  $\mathcal{H}$  through  $I$  and exterior normal vector  $I$ . Clearly,  $\mathcal{H}$  separates  $\mathcal{K}$  and  $\mathcal{D}$  and thus, a fortiori,  $\mathcal{E}(\subset \mathcal{K})$  and  $\mathcal{D}$ . Since  $\mathcal{D}$  is strictly convex by (3),  $\mathcal{D} \cap \mathcal{E} = \{I\}$ . Hence  $B^d$  is the unique ellipsoid of maximum volume in  $C$ .

## Outline of the Proof of Theorem 2

The proof of Theorem 2 is almost identical with that of Theorem 1: an ellipsoid now has the form  $AB^d + a$  and is represented by  $(A, a) \in \mathcal{P} \times \mathbb{E}^d \subset \mathbb{E}^{\frac{1}{2}d(d+3)}$ .  $\mathcal{E}$  is the set

$$\{(A, a) \in \mathcal{P} \times \mathbb{E}^d : A \cdot u \otimes v + a \cdot v \leq h_C(v) \text{ for } u, v \in S^{d-1}\}$$

and instead of (5) we have

$\mathcal{D} \times \mathbb{E}^d$  and  $\mathcal{E}$  are convex,  $(\mathcal{D} \times \mathbb{E}^d) \cap \mathcal{E} = \{(I, o)\}$  and  $\mathcal{D} \times \mathbb{E}^d$  and  $\mathcal{E}$  are separated by the hyperplane  $\mathcal{H} \times \mathbb{E}^d$ , where  $\mathcal{H}$  is the unique support hyperplane of  $\mathcal{D}$  at  $I$  (in  $\mathbb{E}^{\frac{1}{2}d(d+1)}$ ).

$\mathcal{K}$  and  $\mathcal{N}$  are the cones

$$\mathcal{K} = \bigcap_{u \in B^d \cap \text{bd } C} (A, a) \in \mathbb{E}^{\frac{1}{2}d(d+3)} : A \cdot u \otimes u + a \cdot u \leq 1\},$$

$$\mathcal{N} = \text{pos} \{(u \otimes u, u) : u \in B^d \cap \text{bd } C\}.$$

As before,  $(I, o) \in \mathcal{N}$ . Carathéodory's theorem for cones in  $\mathbb{E}^{\frac{1}{2}d(d+3)}$  then shows the following: there are  $(u_k \otimes u_k, u_k) \in \mathcal{N}$  or, equivalently,  $u_k \in B^d \cap \text{bd } C$  and  $\lambda_k > 0, k = 1, \dots, n$ , where  $n \leq \frac{1}{2}d(d+3)$ , such that instead of (8) we have

$$(I, o) = \left( \sum_k \lambda_k u_k \otimes u_k, \sum_k \lambda_k u_k \right).$$

Since  $o = \sum \lambda_k u_k$  and  $\lambda_k > 0$ , the proof that  $n \geq d+1$  is the same as that for  $n \geq d$  above. This concludes the proof that (i) $\Rightarrow$ (ii). The proof of (ii) $\Rightarrow$ (i) is almost the same as that of the corresponding part of the proof of Theorem 1.

### Proof of Theorem 3

Identify a  $d \times d$  matrix  $M = (m_{ij})$  with the point  $(m_{11}, \dots, m_{1d}, m_{21}, \dots, m_{2d}, \dots, m_{dd})^T \in \mathbb{E}^{d^2}$ . The set  $\mathcal{P}'$  of all non-singular  $d \times d$  matrices then is (represented by) an open cone in  $\mathbb{E}^{d^2}$  with apex at the origin. The set

$\mathcal{D}' = \{M \in \mathcal{P}' : |\det M| \geq 1\}$  is a closed body in  $\mathcal{P}'$ , i.e. it is the closure of its interior, with a smooth boundary surface.

The set of all affine images of  $B$  in  $C$  is represented by the set

$$\mathcal{E}' = \{(M, a) \in \mathcal{P}' \times \mathbb{E}^d : Mu \cdot v + a \cdot v = M \cdot u \otimes v + a \cdot v \leq h_C(v) \text{ for } u \in B, v \in S^{d-1}\}.$$

This set is the intersection of the closed halfspaces

$$(9) \quad \{(M, a) \in \mathbb{E}^{d(d+1)} : M \cdot u \otimes v + a \cdot v \leq h_C(v)\} : u \in B, v \in S^{d-1},$$

and thus of a convex set, with the set  $\mathcal{P}' \times \mathbb{E}^d$ . Choose a convex neighborhood  $\mathcal{U}'$  of  $(I, o) \in \mathcal{E}' \cap (\mathcal{P}' \times \mathbb{E}^d)$  which is so small that it is contained in the open set  $\mathcal{P}' \times \mathbb{E}^d$ . By (i),

the convex set  $\mathcal{E}' \cap \mathcal{U}'$  and the smooth body  $\mathcal{D}' \times \mathbb{E}^d$  only have boundary points in common, one being  $(I, o)$ .

Hence  $\mathcal{E}' \cap \mathcal{U}'$  and thus the support cone  $\mathcal{K}'$  of  $\mathcal{E}' \cap \mathcal{U}'$  at  $(I, o)$  is contained in the closed halfspace whose boundary hyperplane is the tangent hyperplane of the smooth body  $\mathcal{D}' \times \mathbb{E}^d$  at  $(I, o)$  and with exterior normal pointing into  $\mathcal{D}' \times \mathbb{E}^d$ . This normal is  $(I, o)$ . The normal cone  $\mathcal{N}'$  of  $\mathcal{K}'$  thus contains  $(I, o)$ .

The support cone  $\mathcal{K}'$  is the intersection of those halfspaces in (9), which contain the apex  $(I, o)$  on their boundary hyperplanes. Thus  $I \cdot u \otimes v + o \cdot v = h_C(v)$ , which is equivalent to  $u \in B \cap \text{bd } C, v \in N_C(u)$ . Hence, these halfspaces have the form

$$\{(M, a) \in \mathbb{E}^{d(d+1)} : M \cdot u \otimes v + a \cdot v \leq h_C(v)\} : u \in B \cap \text{bd } C, v \in N_C(u),$$

where  $N_C(u)$  is the normal cone of  $C$  at the boundary point  $u$ . Thus, being the normal cone of  $\mathcal{K}'$ ,

$$\mathcal{N}' = \text{pos} \{(u \otimes v, v) : u \in B \cap \text{bd } C, v \in N_C(u)\}.$$

Since  $(I, o) \in \mathcal{N}'$ , Carathéodory's theorem for convex cones in  $\mathbb{E}^{d(d+1)}$  yields the following: there are  $(u_k \otimes v_k, v_k) \in \mathcal{N}'$  or, equivalently,  $u_k \in B \cap \text{bd } C, v_k \in N_C(u_k)$ , and  $\lambda_k > 0, k = 1, \dots, n$ , where  $n \leq d(d+1)$ , such that

$$(I, o) = \left( \sum_k \lambda_k u_k \otimes v_k, \sum_k \lambda_k v_k \right).$$

For the proof that  $n \geq d+1$  we show by contradiction that  $\text{lin}\{v_1, \dots, v_n\} = \mathbb{E}^d$  as in the proof of Theorem 1.

## Final Remarks

In different versions of the proofs of Theorems 1 and 2, which are closer to Voronoi's idea, ellipsoids are represented in the form  $x^T A x \leq 1$  and  $(x-a)^T A(x-a) \leq 1$ , respectively.

If in Theorem 3 claim (ii) holds, then the support cone  $\mathcal{K}'$  of  $\mathcal{E}' \cap \mathcal{U}'$  at  $(I, o)$  is contained in the halfspace whose boundary is the tangent hyperplane of  $\mathcal{D}' \times \mathbb{E}^d$  at  $(I, o)$  and with exterior normal pointing into  $\mathcal{D}' \times \mathbb{E}^d$ . Since  $\mathcal{D}' \times \mathbb{E}^d$  is not convex, this does *not* guarantee that  $\mathcal{D}' \times \mathbb{E}^d$  and  $\mathcal{E}'$  do not overlap, i.e. that (i) holds.

**Acknowledgements.** The work of the second author was supported by the Austrian Science Fund (FWF). For his helpful remarks we are obliged to Professor Paul Goodey.

## References

- [1] Ball, K.M., Ellipsoids of maximal volume in convex bodies, *Geom. Dedicata* **41** (1992) 241–250
- [2] Ball, K., Convex geometry and functional analysis, in: *Handbook of the geometry of Banach spaces I* 161–194, North-Holland, Amsterdam 2001
- [3] Bastero, J., Romance, M., John's decomposition of the identity in the non-convex case, *Positivity* **6** (2002) 1–16
- [4] Erdős, P., Gruber, P.M., Hammer, J., *Lattice points*, Longman Scientific, Harlow, Essex, J. Wiley, New York 1989
- [5] Gel'fand, I.M., *Lectures on linear algebra*, Interscience, New York 1967
- [6] Giannopoulos, A.A., Milman, V.D., Euclidean structure in finite dimensional normed spaces, in: *Handbook of the geometry of Banach spaces I* 707–779, North-Holland, Amsterdam 2001
- [7] Giannopoulos, A.A., Perissinaki, I., Tsolomitis, A., John's theorem for an arbitrary pair of convex bodies, *Geom. Dedicata* **84** (2001) 63–79
- [8] Gruber, P.M., Minimal ellipsoids and their duals, *Rend. Circ. Mat. Palermo (2)* **37** (1988) 35–64
- [9] Gruber, P.M., *Convex and discrete geometry*, in preparation
- [10] John, F., Extremum problems with inequalities as subsidiary conditions, in: *Studies and essays presented to R. Courant on his 60th birthday, January 8, 1948*, 187–204, Interscience, New York 1948
- [11] Johnson, W.B., Lindenstrauss, J., Basic concepts in the geometry of Banach spaces, in: *Handbook of the geometry of Banach spaces I* 1–84, North-Holland, Amsterdam 2001
- [12] Pełczyński, A., Remarks on John's theorem on the ellipsoid of maximal volume inscribed into a convex symmetric body in  $\mathbf{R}^n$ , *Note Mat.* **10** (1990), suppl. 2, 395–410
- [13] Roberts, A.W., Varberg, D.E., *Convex functions*, Academic Press, New York 1973
- [14] Ryskov, S.S., Baranovskii, E.P., Classical methods of the theory of lattice packings, *Uspekhi Mat. Nauk* **34** (1979) 3–63, 256
- [15] Schneider, R., *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, Cambridge 1993
- [16] Tomczak-Jaegermann, N., *Banach-Mazur distances and finite-dimensional operator ideals*, Longman Scientific, Harlow, Essex, J. Wiley, New York 1989

Forschungsgruppe  
Konvexe und Diskrete Geometrie  
Technische Universität Wien  
Wiedner Hauptstraße 8–10/1046  
A–1040 Vienna, Austria  
[peter.gruber@tuwien.ac.at](mailto:peter.gruber@tuwien.ac.at)  
[franz.schuster@tuwien.ac.at](mailto:franz.schuster@tuwien.ac.at)